

Asymptotic structure of the Einstein-Maxwell theory on AdS_3

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ABSTRACT: The asymptotic structure of AdS spacetimes in the context of General Relativity coupled to the Maxwell field in three spacetime dimensions is analyzed. Although the fall-off of the fields is relaxed with respect to that of Brown and Henneaux, the variation of the canonical generators associated to the asymptotic Killing vectors can be shown to be finite once required to span the Lie derivative of the fields. The corresponding surface integrals then acquire explicit contributions from the electromagnetic field, and become well-defined provided they fulfill suitable integrability conditions, implying that the leading terms of the asymptotic form of the electromagnetic field are functionally related. Consequently, for a generic choice of boundary conditions, the asymptotic symmetries are broken down to $\mathbb{R} \otimes \text{U}(1) \otimes \text{U}(1)$. Nonetheless, requiring compatibility of the boundary conditions with one of the asymptotic Virasoro symmetries, singles out the set to be characterized by an arbitrary function of a single variable, whose precise form depends on the choice of the chiral copy. Remarkably, requiring the asymptotic symmetries to contain the full conformal group selects a very special set of boundary conditions that is labeled by a unique constant parameter, so that the algebra of the canonical generators is given by the direct sum of two copies of the Virasoro algebra with the standard central extension and $\text{U}(1)$. This special set of boundary conditions makes the energy spectrum of electrically charged rotating black holes to be well-behaved.

KEYWORDS: Black Holes, Classical Theories of Gravity, Gauge-gravity correspondence

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1 Introduction

The very first exact black hole solution endowed with a matter field was the one found by Reissner and Nordström [1, 2]. The simple and remarkable properties that possesses have been widely explored (see, e.g., [3–5] and references therein), which appears to be a natural consequence of the interaction of two of the greatest classical field theories ever formulated: General Relativity and Electromagnetism. The Reissner-Nordström solution can also be generalized to incorporate rotation [6, 7] even if the Einstein-Maxwell theory includes a cosmological constant in diverse dimensions. However, in three-dimensional spacetimes, the analysis of the properties of the corresponding electrically charged (rotating) black hole solution [8–10] can be somewhat darkened due to the logarithmic fall-off of the gauge field, which severely modifies the asymptotic behaviour of the metric. Indeed, finding a suitable definition of global conserved charges in this context turns out to be a very subtle task [10, 11]. Nonetheless, in the case of stationary spherically symmetric solutions, it has been recently shown that the canonical generators associated to the mass and the angular momentum can be naturally defined without the need of any kind of regularization [12]. Furthermore, unlike the higher-dimensional case, the associated surface integrals were found to acquire nontrivial contributions from the electromagnetic field, and consequently, they become manifestly sensitive to the choice of boundary conditions.¹ This

¹We would like to emphasize that hereafter we make a distinction in what we mean by “asymptotic conditions” and “boundary conditions”. For the former we refer to the fall-off of the fields in the asymptotic region, which is an open set; while for the latter, we mean the conditions that are held fixed at the boundary.

effect seems to reconcile the different results that have been obtained for the mass of electrically charged black holes following distinct regularization procedures [10, 13–23], since they might correspond to inequivalent choices of boundary conditions. It is then interesting to explore whether the effect extends to generic configurations, which could be neither static nor spherically symmetric. With this aim, in the next section we address some of the relevant geometric aspects of asymptotically AdS_3 spacetimes in the Einstein-Maxwell theory. They are shown to possess a relaxed behaviour compared with the standard one of Brown and Henneaux [24]. The canonical realization of the asymptotic symmetries is then performed in section 3, where it is shown that the variation of the generators associated to the asymptotic Killing vectors becomes automatically finite once they are required to span the Lie derivative of the fields. The surface integrals generically acquire contributions from the electromagnetic field, and they are subject to nontrivial integrability conditions that imply a functional relationship between the leading terms of the asymptotic form of the electromagnetic field. Since the canonical generators are shown to explicitly depend on the choice of boundary conditions, in section 4 we study the compatibility of different choices with the asymptotic symmetries. It is shown that for generic boundary conditions the asymptotic symmetries are broken down to $\mathbb{R} \otimes \text{U}(1) \otimes \text{U}(1)$. However, there are two different sets of boundary conditions for which only one of the chiral copies of the asymptotic Virasoro symmetries is preserved. It is worth highlighting that requiring preservation of the full conformal group singles out a very special set of boundary conditions that is labeled by unique fixed constant parameter. The canonical realization of the asymptotic symmetry algebra is then given by the direct sum of $\text{U}(1)$ and two copies of the Virasoro algebra with the standard central extension. Section 5 is devoted to compute the global charges of electrically charged rotating black holes, which are found to agree with the expressions that were recently obtained in [12] from the analysis of stationary spherically symmetric configurations. We conclude with some additional remarks in section 6.

2 Fall-off of the fields and asymptotic symmetries

The asymptotic behaviour of the fields can be obtained following the criteria described in [25–28]. For a generic theory, the requirements to be fulfilled can be spelled out as follows:

- The set has to contain as many asymptotic symmetries as possible.
- The fall-off of the fields must be relaxed enough in order to accommodate the solutions of physical interest.
- Simultaneously, the decay has to be sufficiently fast so as to ensure that the variation of the global charges is finite.
- The boundary conditions must guarantee that the variation of the charges can be integrated.

Once the four requirements aforementioned are taken into account for asymptotically AdS_3 spacetimes in the Einstein-Maxwell theory, one is led to propose the following fall-off for

the metric and the gauge field:

$$\begin{aligned}
 g_{\pm\pm} &= \frac{\kappa l^2}{4\pi^2} q_{\pm}^2 \log\left(\frac{r}{l}\right) + f_{\pm\pm} + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-1}\right), \\
 g_{+-} &= -\frac{r^2}{2} + f_{+-} + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-1}\right), \\
 g_{rr} &= \frac{l^2}{r^2} + \frac{f_{rr}}{r^4} + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-5}\right), \\
 g_{r\pm} &= \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-3}\right),
 \end{aligned} \tag{2.1}$$

$$\begin{aligned}
 A_{\pm} &= -\frac{l}{2\pi} q_{\pm} \log\left(\frac{r}{l}\right) + \varphi_{\pm} + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-2}\right), \\
 A_r &= \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-3}\right),
 \end{aligned} \tag{2.2}$$

where $f_{\pm\pm}$, f_{+-} , f_{rr} and q_{\pm} stand for independent arbitrary functions of the null coordinates $x^{\pm} = \frac{t}{l} \pm \phi$. It must be emphasized that the functions φ_{\pm} turn out to be functionally related with q_{\pm} in a precise way. As explained in section 3, this has to be so in order to ensure integrability of the variation of the global charges.

Note that the asymptotic behaviour of the metric in (2.1) might have included additional terms of the form $\mathcal{O}(\log(r/l))$ and $\mathcal{O}(\log(r/l) r^{-4})$, in g_{+-} and g_{rr} respectively, which we do not consider because they can be consistently gauged away. Indeed, for this reason, the asymptotically AdS₃ fall-off for the branch of stationary and spherically symmetric spacetimes studied in [12] fits within the asymptotic structure described by (2.1) and (2.2). Consequently, the asymptotic behaviour proposed here not only accommodates a wide set of exact solutions possessing these symmetries [8, 10, 13, 16, 29–36], but also a larger class of configurations being neither stationary nor spherically symmetric.

The asymptotic symmetries correspond to the subset of diffeomorphisms and local U(1) gauge symmetries, parametrized by $\xi^{\mu} = \xi^{\mu}(x^{\nu})$ and $\eta = \eta(x^{\mu})$ respectively, that preserve the asymptotic behaviour of the fields, i.e., they have to fulfill

$$\begin{aligned}
 \delta_{\xi,\eta} g_{\mu\nu} &= \mathcal{L}_{\xi} g_{\mu\nu} = \mathcal{O}(g_{\mu\nu}), \\
 \delta_{\xi,\eta} A_{\mu} &= \mathcal{L}_{\xi} A_{\mu} + \partial_{\mu} \eta = \mathcal{O}(A_{\mu}).
 \end{aligned} \tag{2.3}$$

By virtue of (2.1) and (2.2) the asymptotic Killing vectors and the asymptotic form of the U(1) parameter are then found to be given by

$$\begin{aligned}
 \xi^{\pm} &= T^{\pm} + \frac{l^2}{2r^2} \partial_{\mp}^2 T^{\mp} + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-4}\right), \\
 \xi^r &= -\frac{r}{2} (\partial_+ T^+ + \partial_- T^-) + \mathcal{O}(r^{-1}), \\
 \eta &= \lambda + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-2}\right),
 \end{aligned} \tag{2.4}$$

respectively, being described by three arbitrary functions $T^{\pm} = T^{\pm}(x^{\pm})$ and $\lambda = \lambda(x^+, x^-)$. Note that the subleading terms of the asymptotic Killing vectors ξ^{\pm} in (2.4) decay slower than that of Brown and Henneaux [24].

It must be highlighted that only a subset of the asymptotic symmetries spanned by (2.4) are associated to canonical generators. This is explained below.

3 Canonical structure

Let us consider the Einstein-Maxwell action in three spacetime dimensions, given by

$$I[g_{\mu\nu}, A_\mu] = \int d^3x \sqrt{-g} \left[\frac{1}{2\kappa} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (3.1)$$

where κ and Λ are related to the Newton constant and the AdS radius according to $\kappa = 8\pi G$ and $\Lambda = -l^{-2}$, respectively. Assuming the standard ADM decomposition of the fields (see, e.g., [37, 38]), up to a boundary term, the Hamiltonian form of the action reads

$$I = \int d^3x \left\{ \pi^{ij} \dot{\gamma}_{ij} + p^i \dot{A}_i - N^\perp \mathcal{H}_\perp - N^i \mathcal{H}_i - \eta \mathcal{G} \right\}, \quad (3.2)$$

where the constraints are given by

$$\begin{aligned} \mathcal{H}_\perp &= \frac{2\kappa}{\sqrt{\gamma}} (\pi^{ij} \pi_{ij} - \pi^2) - \frac{\sqrt{\gamma}}{2\kappa} \left({}^{(2)}R - 2\Lambda \right) + \frac{1}{2\sqrt{\gamma}} \gamma_{ij} p^i p^j + \frac{1}{4} \sqrt{\gamma} F_{ij} F^{ij}, \\ \mathcal{H}_i &= -2\nabla_j \pi_i^j + p^j F_{ij}, \\ \mathcal{G} &= -\partial_i p^i. \end{aligned} \quad (3.3)$$

Here π^{ij} and p^i stand for the momenta associated to the dynamical fields γ_{ij} and A_i , respectively. The smeared total Hamiltonian then naturally splits as

$$\begin{aligned} H(\xi, \eta) &= \int d^3x \left[\epsilon^\perp \mathcal{H}_\perp + \epsilon^i \mathcal{H}_i + \eta \mathcal{G} \right], \\ &= \mathcal{H}(\xi) + \mathcal{G}(\eta), \end{aligned} \quad (3.4)$$

with

$$\epsilon^\perp = N^\perp \xi^t; \quad \epsilon^i = \xi^i + N^i \xi^t. \quad (3.5)$$

It should be emphasized that the generator $\mathcal{H}(\xi)$ does not span the Lie derivative of the gauge field, but instead, a “pure diffeomorphism” given by $\delta_\xi A_\mu = \{A_\mu, \mathcal{H}(\xi)\} = \xi^\nu F_{\nu\mu}$. Hence, in order to make contact with the asymptotic symmetries in (2.3), the generator has to be “improved” by the addition of a suitable U(1) gauge transformation. The improved generator is obtained from (3.4) provided the parameter of the U(1) gauge transformation is of the form $\eta = \lambda + \xi^\mu A_\mu$ (see, e.g., [39, 40]), so that the total Hamiltonian now splits in a different way, according to

$$H(\xi, \lambda) = \tilde{\mathcal{H}}(\xi) + \mathcal{G}(\lambda), \quad (3.6)$$

where

$$\tilde{\mathcal{H}}(\xi) = \mathcal{H}(\xi) + \mathcal{G}(\xi^\mu A_\mu), \quad (3.7)$$

stands for the improved generator that fulfills what we were looking for, i.e.,

$$\{A_\mu, H(\xi, \lambda)\} = \{A_\mu, \tilde{\mathcal{H}}(\xi) + \mathcal{G}(\lambda)\} = \mathcal{L}_\xi A_\mu + \partial_\mu \lambda, \quad (3.8)$$

on-shell.

Remarkably, once the improved generator $\tilde{\mathcal{H}}(\xi)$ is supplemented by the appropriate boundary term that makes it to be well-defined everywhere [41], the variation of the corresponding surface integrals becomes automatically finite. Indeed, the surface integral associated to $\mathcal{G}(\xi^\mu A_\mu)$ precisely cancels out the divergences of the surface integral that corresponds to the pure diffeomorphisms generator $\mathcal{H}(\xi)$. It should be highlighted that this effect occurs due to the slow fall-off of the electromagnetic field in three spacetime dimensions, so that the improvement term amounts to an improper gauge transformation that regularizes the global charges. Note that this is not the case for asymptotically AdS spacetimes in $d \geq 4$ dimensions, because the improvement term in (3.7) just corresponds to a proper gauge transformation that does not modify the surface integrals associated to the canonical generators. This is explicitly shown in what follows.

3.1 Finite conserved charges and their integrability conditions

The variation of the boundary terms which ensure that the canonical generators $\tilde{\mathcal{H}}(\xi)$ and $\mathcal{G}(\lambda)$ are well-defined is denoted by $\delta Q_{\tilde{\mathcal{H}}}(\xi)$ and $\delta Q_{\mathcal{G}}(\lambda)$, respectively. By virtue of the fall-off in (2.1), (2.2), the variation of the U(1) generator is finite and readily integrates as

$$Q_{\mathcal{G}}(\lambda) = \int dS_l \lambda p^l = \frac{1}{2\pi} \int d\phi \lambda (q_+ + q_-) = \frac{1}{2\pi} \int d\phi \lambda q_t. \quad (3.9)$$

The variation of the surface integral associated to the improved generator, given by $\delta Q_{\tilde{\mathcal{H}}}(\xi)$, is also finite, but becomes subject to nontrivial integrability conditions.

In order to see how the regularization mechanism is intrinsically built in, following eq. (3.7), it is instructive to split the variation of the surface integral according to

$$\delta Q_{\tilde{\mathcal{H}}}(\xi) = \delta Q_{\mathcal{H}}(\xi) + \delta Q_{\mathcal{G}}(\xi^\mu A_\mu), \quad (3.10)$$

with

$$\begin{aligned} \delta Q_{\mathcal{H}}(\xi) = \int dS_l \left[\frac{1}{2\kappa} \epsilon^\perp G^{ijkl} \nabla_k \delta \gamma_{ij} - \frac{1}{2\kappa} \nabla_k \epsilon^\perp G^{ijkl} \delta \gamma_{ij} + 2\epsilon^j \delta \left(\gamma_{jk} \pi^{kl} \right) \right. \\ \left. - \epsilon^l \pi^{jk} \delta \gamma_{jk} - \epsilon^\perp \sqrt{\gamma} F^{li} \delta A_i - \left(\epsilon^l p^i - \epsilon^i p^l \right) \delta A_i \right], \end{aligned} \quad (3.11)$$

$$\delta Q_{\mathcal{G}}(\xi^\mu A_\mu) = \int dS_l \xi^\mu A_\mu \delta p^l,$$

where $G^{ijkl} = \frac{1}{2} \gamma^{1/2} (\gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} - 2\gamma^{ij} \gamma^{kl})$.

Making use of the asymptotic fall-off of the fields (2.1), (2.2), as well as the asymptotic Killing vectors (2.4), the variation of the surface integrals reduces to

$$\begin{aligned} \delta Q_{\mathcal{G}}(T^+, T^-) &= \delta Q_{\mathcal{G}}^+(T^+) + \delta Q_{\mathcal{G}}^-(T^-), \\ \delta Q_{\mathcal{H}}(T^+, T^-) &= \delta Q_{\mathcal{H}}^+(T^+) + \delta Q_{\mathcal{H}}^-(T^-), \end{aligned} \quad (3.12)$$

with

$$\begin{aligned}\delta Q_{\mathcal{G}}^{\pm}(T^{\pm}) &= -\frac{1}{2\pi} \int d\phi T^{\pm} \left[\frac{l}{2\pi} (q_{\pm} \delta q_{+} + q_{\pm} \delta q_{-}) \log\left(\frac{r}{l}\right) - \varphi_{\pm} (\delta q_{+} + \delta q_{-}) \right], \\ \delta Q_{\mathcal{H}}^{\pm}(T^{\pm}) &= \frac{1}{2\pi} \int d\phi T^{\pm} \left[\frac{l}{2\pi} (q_{\pm} \delta q_{+} + q_{\pm} \delta q_{-}) \log\left(\frac{r}{l}\right) \pm q_{\pm} (\delta \varphi_{+} \mp \delta \varphi_{-}) \right] \\ &\quad + \frac{1}{2\pi} \int d\phi T^{\pm} \left[\delta \left(\frac{2\pi}{l\kappa} f_{\pm\pm} \mp \frac{l}{4\pi} q_{\pm} (q_{+} - q_{-}) \right) \right].\end{aligned}\quad (3.13)$$

Therefore, it is clear that the logarithmic divergences in (3.13) precisely cancel out, so that the variation of the improved generator (3.10) reads

$$\delta Q_{\tilde{\mathcal{H}}}(T^{+}, T^{-}) = \delta Q_{\mathcal{H}}^{+}(T^{+}) + \delta Q_{\mathcal{H}}^{-}(T^{-}) = \int d\phi T^{+} \delta \mathcal{L}_{+} + \int d\phi T^{-} \delta \mathcal{L}_{-}, \quad (3.14)$$

where

$$\delta \mathcal{L}_{\pm} = \frac{1}{2\pi} \delta \left(\frac{2\pi}{l\kappa} f_{\pm\pm} \mp \frac{l}{4\pi} q_{\pm} (q_{+} - q_{-}) \pm q_{\pm} (\varphi_{+} - \varphi_{-}) \right) + \frac{1}{2\pi} [\varphi_{+} \delta q_{-} + \varphi_{-} \delta q_{+}]. \quad (3.15)$$

Note that the terms within the square brackets in (3.15) lead to nontrivial integrability conditions of the form

$$\delta^2 Q_{\tilde{\mathcal{H}}}^{\pm} = \frac{1}{2\pi} \int d\phi T^{\pm} [\delta \varphi_{+} \wedge \delta q_{-} + \delta \varphi_{-} \wedge \delta q_{+}] = 0, \quad (3.16)$$

which implies that φ_{\pm} and q_{\pm} are functionally related. Assuming that q_{+} and q_{-} vary independently, the integrability conditions are solved by any arbitrary function $\mathcal{V} = \mathcal{V}(q_{+}, q_{-})$ that fulfills

$$\varphi_{\pm} = \frac{1}{2} \frac{\delta \mathcal{V}}{\delta q_{\mp}}. \quad (3.17)$$

Hence, the surface integrals associated to the improved canonical generators integrate as

$$Q_{\tilde{\mathcal{H}}}^{\pm}(T^{\pm}) = \int d\phi T^{\pm} \mathcal{L}_{\pm}, \quad (3.18)$$

with

$$\mathcal{L}_{\pm} = \frac{1}{l\kappa} f_{\pm\pm} \mp \frac{lq_{\pm}}{8\pi^2} \left[q_{+} - q_{-} - \frac{2\pi}{l} \left(\frac{\delta \mathcal{V}}{\delta q_{-}} - \frac{\delta \mathcal{V}}{\delta q_{+}} \right) \right] + \frac{1}{4\pi} \mathcal{V}. \quad (3.19)$$

Note that the surface integrals manifestly acquire contributions coming from the electromagnetic field, and moreover, they also explicitly depend on the arbitrary function \mathcal{V} that has to be specified by the boundary conditions in order to guarantee the integrability of the global charges. This is a crucial point because otherwise, if the variation of the canonical generators were not integrable, the whole canonical structure would be spoiled, since it would conflict with the fact that the Poisson brackets fulfill the Jacobi identity.

It must also be emphasized that fulfilling the integrability conditions implies that only a subset of the purely geometric asymptotic symmetries in (2.4) are reflected in the canonical realization. Indeed, the amount of asymptotic symmetries that are spanned by canonical generators depends on the choice of boundary conditions specified by \mathcal{V} . This is explained in the next section.

4 Compatibility of the boundary conditions with the asymptotic symmetries

The boundary conditions are characterized by an arbitrary function \mathcal{V} that, according to (3.17), specifies the functional relationship of the leading terms of the asymptotic form of the electromagnetic gauge field, i.e., $\varphi_{\pm} = \varphi_{\pm}(q_+, q_-)$. Consequently, a generic choice of boundary conditions turns out to be generically incompatible with the whole set of asymptotic symmetries in (2.4), because the transformations rules of φ_{\pm} and q_{\pm} have to be consistent with

$$\delta\varphi_{\pm} = \frac{\delta\varphi_{\pm}}{\delta q_+} \delta q_+ + \frac{\delta\varphi_{\pm}}{\delta q_-} \delta q_- . \quad (4.1)$$

In what follows this is first analyzed for the asymptotic U(1) gauge transformations and then for the asymptotic Killing vectors.

4.1 Asymptotic U(1) gauge transformations

In the case of asymptotic U(1) gauge transformations spanned by (2.4) with $T^{\pm} = 0$, the functions φ_{\pm} and q_{\pm} transform according to

$$\delta\varphi_{\pm} = \partial_{\pm}\lambda ; \quad \delta q_{\pm} = 0 . \quad (4.2)$$

Therefore, eq. (4.1) can only be compatible with the transformation law in (4.2) provided $\partial_{\pm}\lambda = 0$.

One then concludes that, if q_+ and q_- are assumed to vary independently, only the zero mode of λ turns out to be consistent, regardless the choice of boundary conditions. Hence, there is just a single global charge that is conserved, which corresponds to (3.9) evaluated for $\lambda = \lambda_0$ constant. This is the electric charge \mathcal{Q} .

Alternatively, the absence of additional conserved currents can be easily verified from the conservation of the surface integral in (3.9). Indeed, making use of the leading term of the Maxwell equation in the asymptotic region, given by $\partial_+ q_- + \partial_- q_+ = \partial_{\phi} q_{\phi} - l \partial_t q_t = 0$, requiring $\dot{Q}_{\mathcal{G}}(\lambda) = 0$, implies that λ asymptotically approaches to a constant. One can then get rid off the constant U(1) parameter so that (3.9) agrees with the Gauss formula.

It is worth pointing out that, as the remaining modes of λ do not yield to conserved charges, they can not be regarded as legitimate asymptotic symmetries, since they do not possess canonical generators.²

4.2 Asymptotic Killing vectors

Under the action of a generic asymptotic Killing vector spanned by (2.4) with $\lambda = 0$, the transformation laws of φ_{\pm} and q_{\pm} are given by

$$\begin{aligned} \delta\varphi_{\pm} &= \partial_{\pm}(\varphi_{\pm} T^{\pm}) + T^{\mp} \partial_{\mp} \varphi_{\pm} + \frac{\ell}{4\pi} q_{\pm} (\partial_+ T^+ + \partial_- T^-) , \\ \delta q_{\pm} &= \partial_{\pm}(q_{\pm} T^{\pm}) + T^{\mp} \partial_{\mp} q_{\pm} . \end{aligned} \quad (4.3)$$

²Note that additional U(1) currents could be consistent, but for different classes of boundary conditions that are not considered in this work. Indeed, for instance, this would be the case if either q_+ or q_- is set to vanish without variation, so that the modes of $Q_{\mathcal{G}}(\lambda)$ correspond to the generators of left or right currents.

In the case of (right) asymptotic symmetries spanned by T^+ (with $T^- = 0$), compatibility of the transformation law in (4.3) with eq. (4.1) implies the following conditions:

$$\begin{aligned} \left(\frac{\ell}{4\pi} q_+ + \frac{1}{2} \frac{\delta \mathcal{V}}{\delta q_-} - \frac{1}{2} q_+ \frac{\delta^2 \mathcal{V}}{\delta q_+ \delta q_-} \right) \partial_+ T^+ &= 0, \\ \left(\frac{\ell}{4\pi} q_- - \frac{1}{2} q_+ \frac{\delta^2 \mathcal{V}}{\delta q_+^2} \right) \partial_+ T^+ &= 0. \end{aligned} \quad (4.4)$$

Hence, for a generic choice of \mathcal{V} , eqs. (4.4) can only be fulfilled provided $\partial_+ T^+ = 0$, i.e., just for the zero mode of T^+ .

Requiring the whole set of modes of T^+ to be compatible with the conditions in (4.4) implies that the function \mathcal{V} has to satisfy both equations within the round brackets in (4.4). Interestingly, up to an additive constant, this singles out the set of boundary conditions to be characterized by

$$\mathcal{V} = \mathcal{V}^+ = \frac{l q_+ q_-}{2\pi} [\log(q_+) - 1] + q_+ \mathcal{C}_+(q_-), \quad (4.5)$$

where \mathcal{C}_+ is an arbitrary function of a single variable.

If the asymptotic symmetries correspond to the ones spanned by T^- , compatibility of (4.3) with (4.1) implies that

$$\begin{aligned} \left(\frac{\ell}{4\pi} q_- + \frac{1}{2} \frac{\delta \mathcal{V}}{\delta q_+} - \frac{1}{2} q_- \frac{\delta^2 \mathcal{V}}{\delta q_+ \delta q_-} \right) \partial_- T^- &= 0, \\ \left(\frac{\ell}{4\pi} q_+ - \frac{1}{2} q_- \frac{\delta^2 \mathcal{V}}{\delta q_-^2} \right) \partial_- T^- &= 0. \end{aligned} \quad (4.6)$$

Analogously, for generic boundary conditions specified by \mathcal{V} , both eqs. in (4.6) can only be satisfied for the zero mode of T^- .

The family of boundary conditions that is consistent with the entire set of modes of T^- then corresponds to a choice of \mathcal{V} that fulfills the differential equations in the round brackets of (4.6). Up to an additive constant, the set is described by an arbitrary function $\mathcal{C}_-(q_+)$, so that

$$\mathcal{V} = \mathcal{V}^- = \frac{l q_+ q_-}{2\pi} [\log(q_-) - 1] + q_- \mathcal{C}_-(q_+). \quad (4.7)$$

Remarkably, one then concludes that the whole set of asymptotic Killing vectors, spanned by T^+ and T^- , turns out to be compatible with the integrability of the canonical generators provided the boundary conditions are specified by a very special set, being labeled by a unique arbitrary constant parameter ζ without variation. This is because simultaneously preserving left and right modes, implies that $\mathcal{V} = \mathcal{V}^+ = \mathcal{V}^-$. Hence, by virtue of (4.5) and (4.7), the very special set of boundary conditions that preserves the full conformal group is determined by the choice

$$\mathcal{V} = \frac{l}{2\pi} q_+ q_- \log(\zeta q_+ q_-). \quad (4.8)$$

This result agrees with the one recently found in [12], which was obtained from a completely different approach. Indeed, the same choice of boundary conditions was recovered, but from requiring compatibility with scaling and Lorentz symmetries of stationary spherically symmetric solutions.

4.3 Algebra of the canonical generators

Following the results explained right above this section, one concludes that the algebra of the canonical generators depends on the choice of the boundary conditions specified by \mathcal{V} .

For a generic choice of \mathcal{V} , the asymptotic symmetries that are compatible with the boundary conditions just correspond to the zero modes of T^+ , T^- , λ in (2.4). Hence, there are only three conserved charges given by (3.9) and (3.18) with constant parameters. Therefore, since the asymptotic symmetries turn out to be spanned by ∂_t , ∂_ϕ and a global $U(1)$ transformation, the asymptotic symmetry group is given by $\mathbb{R} \otimes U(1) \otimes U(1)$, so that the global charges correspond to the mass, the angular momentum and the electric charge.

In the case of the very special choice of boundary conditions given by \mathcal{V} in (4.8), the legitimate asymptotic symmetries that possess well-defined canonical generators correspond to the full conformal group spanned by T^+ , T^- , and the global $U(1)$ transformation generated by the zero mode of λ . The surface integrals associated to the improved canonical generators in (3.18) are such that in this case eq. (3.19) reduces to

$$\mathcal{L}_\pm = \frac{1}{l\kappa} f_{\pm\pm} + \frac{lq_\pm^2}{8\pi^2} \log(\zeta q_+ q_-) . \quad (4.9)$$

It is worth highlighting that the $+r$ and $-r$ components of the Einstein field equations, which generically read

$$\partial_\mp f_{\pm\pm} + \frac{\kappa\ell^2}{8\pi^2} \left[q_\mp \partial_\pm q_\pm \pm \frac{2\pi}{\ell} q_\pm \left(\partial_- \left(\frac{\delta\mathcal{V}}{\delta q_-} \right) - \partial_+ \left(\frac{\delta\mathcal{V}}{\delta q_+} \right) + \frac{3\ell}{2\pi} \partial_+ q_- \right) \right] = 0 , \quad (4.10)$$

in this case just reduce to

$$\partial_\mp \mathcal{L}_\pm = 0 . \quad (4.11)$$

Therefore, since the Poisson brackets fulfill $\{Q(Y_1), Q(Y_2)\} = \delta_{Y_2} Q(Y_1)$, the algebra of the canonical generators can be readily obtained from the transformation law of q_\pm in (4.3), (4.2), as well as from the one of $f_{\pm\pm}$, which reads

$$\delta f_{\pm\pm} = 2f_{\pm\pm} \partial_\pm T^\pm + T^\pm \partial_\pm f_{\pm\pm} - \frac{\ell^2}{2} \partial_\pm^3 T^\pm - \frac{\kappa\ell^2}{8\pi^2} q_\pm^2 (\partial_+ T^\pm + \partial_- T^\mp) + T^\mp \partial_\mp f_{\pm\pm} . \quad (4.12)$$

Hence, the transformation laws of \mathcal{L}_\pm in (4.9) are given by

$$\delta \mathcal{L}_\pm = 2\mathcal{L}_\pm \partial_\pm T^\pm + T^\pm \partial_\pm \mathcal{L}_\pm - \frac{l}{2\kappa} \partial_\pm^3 T^\pm . \quad (4.13)$$

Expanding in Fourier modes, $\mathcal{L}_\pm = \frac{1}{2\pi} \sum_m \mathcal{L}_m^\pm e^{im\phi}$, the algebra of the canonical generators \mathcal{L}_m^\pm with the electric charge \mathcal{Q} then reads

$$\begin{aligned} i \{ \mathcal{L}_m^\pm, \mathcal{L}_n^\pm \} &= (m-n) \mathcal{L}_{m+n}^\pm + \frac{c}{12} m^3 \delta_{m+n} , \\ i \{ \mathcal{L}_m^\pm, \mathcal{Q} \} &= 0 , \\ i \{ \mathcal{Q}, \mathcal{Q} \} &= 0 , \end{aligned} \quad (4.14)$$

with $c = 3l/2G$, corresponding to the direct sum of $U(1)$ and two copies of the Virasoro algebra with the standard central extension.

As a final remark of this section, it is worth mentioning that if the boundary conditions were chosen so that $\mathcal{V} = \mathcal{V}^+$ in (4.5), the legitimate asymptotic symmetries are spanned by T^+ , and the zero modes of T^- , λ . The canonical generators then correspond to \mathcal{L}_m^+ , \mathcal{L}_0^- , and \mathcal{Q} , whose algebra is given by the direct sum of the right copy of the Virasoro algebra with the Brown-Henneaux central extension with two additional Abelian generators. Analogously, for the choice $\mathcal{V} = \mathcal{V}^-$ in (4.7), the same occurs for the left copy.

5 Mass and angular momentum of electrically charged rotating black holes

The rotating extension of the static BTZ black hole with electric charge [8] was independently obtained in [9] and [10] through different approaches. It is convenient to express the solution as in [12], so that the line element and the electromagnetic gauge field can be written as

$$ds^2 = -N^2 F^2 dt^2 + \frac{d\rho^2}{F^2} + R^2 \left(N^\phi dt + d\phi \right)^2, \quad (5.1)$$

$$A = A_t dt + A_\phi d\phi,$$

with

$$\begin{aligned} R^2 &= \rho^2 + \left(\frac{\omega^2}{1 - \omega^2} \right) r_+^2 + \frac{\kappa}{4\pi^2} (q_t \omega l)^2 \log \left(\frac{\rho}{r_+} \right), \\ N^\phi &= - \left(\frac{\omega}{1 - \omega^2} \right) \left(\frac{\rho^2}{l^2} - F^2 \right) \frac{l}{R^2}, \\ N^2 &= \frac{\rho^2}{R^2}, \\ F^2 &= \frac{\rho^2}{l^2} - \frac{r_+^2}{l^2} - \frac{\kappa}{4\pi^2} q_t^2 (1 - \omega^2) \log \left(\frac{\rho}{r_+} \right), \\ A_t &= -\frac{q_t}{2\pi} \log \left(\frac{\rho}{l} \right) + \frac{\varphi_t}{l}, \\ A_\phi &= \frac{q_t \omega l}{2\pi} \log \left(\frac{\rho}{l} \right) + \varphi_\phi, \end{aligned} \quad (5.2)$$

where r_+ , ω , q_t , φ_t , φ_ϕ stand for arbitrary constants. Here, the Lagrange multipliers have been chosen as $N_\infty = 1$, and $N_\infty^\phi = \Phi = 0$.

Note that in the asymptotic region, $\rho \rightarrow \infty$, the metric components g_{+-} and $g_{\rho\rho}$ possess terms that behave as $\mathcal{O}(\log(\rho/l))$ and $\mathcal{O}(\log(\rho/l) \rho^{-4})$, respectively; which as explained in section (2), can be consistently gauged away. Indeed, changing the coordinates according to $x^\pm = \frac{t}{l} \pm \phi$, and $\rho = r + \frac{\kappa l^2}{16\pi^2} q_t^2 (1 - \omega^2) \log \left(\frac{r}{l} \right) r^{-1}$, the asymptotic behaviour of the solution is given by

$$\begin{aligned} g_{\pm\pm} &= \frac{\kappa l^2}{4\pi^2} q_\pm^2 \log \left(\frac{r}{l} \right) + f_{\pm\pm} + \mathcal{O} \left(\log \left(\frac{r}{l} \right) r^{-1} \right), \\ g_{+-} &= -\frac{r^2}{2} + f_{+-} + \mathcal{O} \left(\log \left(\frac{r}{l} \right) r^{-1} \right), \\ g_{rr} &= \frac{l^2}{r^2} + \frac{f_{rr}}{r^4} + \mathcal{O} \left(\log \left(\frac{r}{l} \right) r^{-5} \right), \\ g_{r\pm} &= 0, \end{aligned} \quad (5.3)$$

$$A_{\pm} = -\frac{l}{2\pi} q_{\pm} \log\left(\frac{r}{l}\right) + \varphi_{\pm} + \mathcal{O}\left(\log\left(\frac{r}{l}\right) r^{-2}\right),$$

$$A_r = 0,$$

with

$$\begin{aligned} q_{\pm} &= \frac{1}{2} (q_t \pm q_{\phi}) ; \quad q_{\phi} = -\omega q_t , \\ f_{\pm\pm} &= \left(\frac{1 \mp \omega}{1 \pm \omega}\right) \frac{r_{\pm}^2}{4} - \frac{l^2 q_t^2 \kappa (1 \mp \omega)^2}{16\pi^2} \log\left(\frac{r_{\pm}}{l}\right) , \\ f_{+-} &= \frac{4\pi^2 r_{+}^2 - l^2 q_t^2 \kappa \log\left(\frac{r_{+}}{l}\right) (1 - \omega^2)}{16\pi^2} , \\ f_{rr} &= 8l^2 \pi^2 r_{+}^2 + l^4 q_t^2 \kappa (1 - \omega^2) \left(1 - \log\left(\frac{r_{+}^2}{l^2}\right)\right) , \\ \varphi_{\pm} &= \frac{1}{2} (\varphi_t \pm \varphi_{\phi}) , \end{aligned} \tag{5.4}$$

which perfectly fits within our asymptotic conditions in eqs. (2.1), (2.2). Therefore, the global charges can be readily obtained from the surface integrals in (3.9) and (3.18).

For a generic choice of boundary conditions, specified by an arbitrary function \mathcal{V} , the electric charge is given by $\mathcal{Q} = q_t$, while the remaining charges are determined by \mathcal{L}_{\pm} in eq. (3.19). Hence, the mass and the angular momentum read

$$\begin{aligned} M &= \frac{2\pi}{l} (\mathcal{L}_{+} + \mathcal{L}_{-}) \\ &= \frac{\pi r_{+}^2}{\kappa l^2} \left(\frac{1 + \omega^2}{1 - \omega^2}\right) - \frac{q_t^2}{4\pi} \left[\omega^2 + (1 + \omega^2) \log\left(\frac{r_{+}}{l}\right)\right] + \frac{1}{l} \left(\mathcal{V} - q_{\phi} \frac{\partial \mathcal{V}}{\partial q_{\phi}}\right) , \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} J &= 2\pi (\mathcal{L}_{+} - \mathcal{L}_{-}) \\ &= -\frac{2\pi r_{+}^2 \omega}{\kappa l (1 - \omega^2)} + \frac{l q_t^2 \omega}{4\pi} \left(1 + \log\left(\frac{r_{+}^2}{l^2}\right)\right) - q_t \frac{\delta \mathcal{V}}{\delta q_{\phi}} , \end{aligned} \tag{5.6}$$

respectively, in full agreement with the expressions found in [12] from a minisuperspace of stationary spherically symmetric configurations.³ In the case of boundary conditions with $\mathcal{V} = 0$, the mass and the angular momentum then reduce to the ones found in [10].

For the very special choice of boundary conditions that is compatible with the full conformal symmetry at infinity, with \mathcal{V} specified by (4.8), the zero modes of the Virasoro generators are given by

$$\mathcal{L}_{\pm} = \frac{r_{+}^2}{4l\kappa} \left(\frac{1 \mp \omega}{1 \pm \omega}\right) + \frac{l q_t^2 (1 \mp \omega)^2}{32\pi^2} \left[\log\left(\frac{\kappa}{8\pi^2} \frac{q_t^2 l^2}{r_{+}^2} (1 - \omega^2)\right) + \gamma - 1\right] , \tag{5.7}$$

so that the mass and the angular momentum in this case read

$$M = \frac{\pi}{\kappa} \left(\frac{1 + \omega^2}{1 - \omega^2}\right) \frac{r_{+}^2}{l^2} + \frac{q_t^2 (1 + \omega^2)}{8\pi} \left[\log\left(\frac{\kappa}{8\pi^2} \frac{q_t^2 l^2}{r_{+}^2} (1 - \omega^2)\right) + \gamma - 1\right] , \tag{5.8}$$

³The difference in the sign of the angular momentum is because here we have chosen the opposite orientation as compared with [12].

and

$$J = -\frac{2l\omega}{1+\omega^2}M, \quad (5.9)$$

where the arbitrary fixed constant ζ that characterizes this set of boundary conditions has been redefined as $\gamma = 1 + \log\left(\frac{2\pi^2}{\kappa}\zeta\right)$. Therefore, according to [12], this parameter can be interpreted as the slope that bounds the allowed region in the parameter space where the solution describes a black hole. It is worth pointing out that for this set of boundary conditions, the black hole energy spectrum is nonnegative, and for a fixed value of the mass, the electric charge possesses an upper bound.

6 Ending remarks

Improving the canonical generators as in (3.7), so that they span the Lie derivative of the fields, was shown to be a crucial point in order to unveil the asymptotic structure of electromagnetism coupled to gravity with negative cosmological constant in three spacetime dimensions. Despite the fall-off of the fields is extremely relaxed as compared with the standard one, the global charges were shown to be finite and manifestly acquire contributions from the electromagnetic field. The existence of the canonical structure requires nontrivial integrability conditions for the global charges to be fulfilled, which implies a functional relationship between the leading terms in the asymptotic form of the electromagnetic field. The global charges then also explicitly depend on the choice of boundary conditions. This effect has been also shown to occur for General Relativity coupled to scalar and higher spin fields in three spacetime dimensions [42–45].

In the case of the Maxwell field, the boundary conditions generically break most of the purely geometric asymptotic symmetries, but nonetheless, there is a very special choice that is compatible with the full conformal symmetry at infinity.⁴ It is then clear that the possibility of studying holography along the lines of the AdS₃/CFT₂ correspondence [46–49], becomes widened to the case of a fully backreacting Maxwell field. Moreover, the different possible choices of boundary conditions that are compatible with the integrability of the global charges, might also be of interest in the context of holographic superconductors [50–55].

We would like to recall that our results have been obtained assuming that the functions that characterize the leading terms in the fall-off of the electromagnetic field, given by q_+ and q_- , are assumed to vary independently. This has to be so in order to accommodate the generic electrically charged rotating black hole solution within the asymptotic conditions in (2.1), (2.2). Our set of asymptotically AdS₃ conditions differs from the ones that have been previously explored in the context of the Einstein-Maxwell theory [18, 56]. It is then worth pointing out that further inequivalent sets of boundary conditions could also be constructed if one no longer assumes that both q_+ , q_- are free to vary in an independent way, so that the whole analysis would have to be done from scratch.

⁴An intermediate situation is known to occur for asymptotically AdS₃ spacetimes in topologically massive gravity, where the integrability conditions are even more stringent, so that the boundary conditions preserve at most one chiral copy of the Virasoro algebra, while for the other one only the zero mode survives [28].

As a final remark, it is worth mentioning that the effects of improving the canonical generators according to (3.7) do not show up for asymptotically AdS spacetimes in $d \geq 4$ dimensions. Indeed, in the higher-dimensional case, the fall-off of the electromagnetic field is slow enough so that the metric fulfills the standard asymptotically AdS behaviour [25, 57]. Consequently, since the time and angular components of the asymptotic Killing vectors behave as $\mathcal{O}(1)$ in the asymptotic region, the variation of the improvement term, given by $\int dS_l \xi^\mu A_\mu \delta p^l$, necessarily vanishes when $r \rightarrow \infty$. Hence, the improved and the standard generators of diffeomorphisms just differ by a term that spans a proper gauge transformation which does not contribute to the global charges.

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References

- [1] H. Reissner, *Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie*, *Ann. Phys.* **50** 106 (1916) 120.
- [2] G. Nordström, *On the energy of the gravitational field in Einstein’s theory*, *Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk.* **26** (1918) 1201.
- [3] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge U.K. (1975).
- [4] P.K. Townsend, *Black holes: lecture notes*, [gr-qc/9707012](#) [[INSPIRE](#)].
- [5] L.J. Romans, *Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory*, *Nucl. Phys. B* **383** (1992) 395 [[hep-th/9203018](#)] [[INSPIRE](#)].
- [6] R.P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, *Phys. Rev. Lett.* **11** (1963) 237 [[INSPIRE](#)].
- [7] E.T. Newman et al., *Metric of a rotating, charged mass*, *J. Math. Phys.* **6** (1965) 918.
- [8] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)] [[INSPIRE](#)].

- [9] G. Clement, *Classical solutions in three-dimensional Einstein-Maxwell cosmological gravity*, *Class. Quant. Grav.* **10** (1993) L49 [INSPIRE].
- [10] C. Martinez, C. Teitelboim and J. Zanelli, *Charged rotating black hole in three space-time dimensions*, *Phys. Rev. D* **61** (2000) 104013 [hep-th/9912259] [INSPIRE].
- [11] S. Deser and P.O. Mazur, *Static solutions in $D = 3$ Einstein-Maxwell theory*, *Class. Quant. Grav.* **2** (1985) L51.
- [12] A. Pérez, M. Riquelme, D. Tempo and R. Troncoso, *Conserved charges and black holes in the Einstein-Maxwell theory on AdS_3 reconsidered*, *JHEP* **10** (2015) 161 [arXiv:1509.01750] [INSPIRE].
- [13] G. Clement, *Spinning charged BTZ black holes and selfdual particle-like solutions*, *Phys. Lett. B* **367** (1996) 70 [gr-qc/9510025] [INSPIRE].
- [14] K.C.K. Chan, *Comment on the calculation of the angular momentum for the (anti)selfdual charged spinning BTZ black hole*, *Phys. Lett. B* **373** (1996) 296 [gr-qc/9509032] [INSPIRE].
- [15] M. Kamata and T. Koikawa, *$(2 + 1)$ -dimensional charged black hole with (anti-)selfdual Maxwell fields*, *Phys. Lett. B* **391** (1997) 87 [hep-th/9605114] [INSPIRE].
- [16] O.J.C. Dias and J.P.S. Lemos, *Rotating magnetic solution in three-dimensional Einstein gravity*, *JHEP* **01** (2002) 006 [hep-th/0201058] [INSPIRE].
- [17] G. Clement, *Black hole mass and angular momentum in $2 + 1$ gravity*, *Phys. Rev. D* **68** (2003) 024032 [gr-qc/0301129] [INSPIRE].
- [18] M. Cadoni, M. Melis and M.R. Setare, *Microscopic entropy of the charged BTZ black hole*, *Class. Quant. Grav.* **25** (2008) 195022 [arXiv:0710.3009] [INSPIRE].
- [19] Y.S. Myung, Y.-W. Kim and Y.-J. Park, *Entropy function approach to charged BTZ black hole*, *Gen. Rel. Grav.* **42** (2010) 1919 [arXiv:0903.2109] [INSPIRE].
- [20] K. Jensen, *Chiral anomalies and AdS/CMT in two dimensions*, *JHEP* **01** (2011) 109 [arXiv:1012.4831] [INSPIRE].
- [21] A.A. Garcia-Diaz, *Three dimensional stationary cyclic symmetric Einstein-Maxwell solutions; energy, mass, momentum and algebraic tensors characteristics*, arXiv:1307.6652 [INSPIRE].
- [22] S.H. Hendi, S. Panahiyan and R. Mamasani, *Thermodynamic stability of charged BTZ black holes: ensemble dependency problem and its solution*, *Gen. Rel. Grav.* **47** (2015) 91 [arXiv:1507.08496] [INSPIRE].
- [23] A.M. Frassino, R.B. Mann and J.R. Mureika, *Lower-dimensional black hole chemistry*, *Phys. Rev. D* **92** (2015) 124069 [arXiv:1509.05481] [INSPIRE].
- [24] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207 [INSPIRE].
- [25] M. Henneaux and C. Teitelboim, *Asymptotically Anti-de Sitter spaces*, *Commun. Math. Phys.* **98** (1985) 391 [INSPIRE].
- [26] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, *Asymptotic behavior and Hamiltonian analysis of Anti-de Sitter gravity coupled to scalar fields*, *Ann. Phys.* **322** (2007) 824 [hep-th/0603185] [INSPIRE].
- [27] M. Henneaux, C. Martinez and R. Troncoso, *Asymptotically Anti-de Sitter spacetimes in topologically massive gravity*, *Phys. Rev. D* **79** (2009) 081502 [arXiv:0901.2874] [INSPIRE].

- [28] M. Henneaux, C. Martinez and R. Troncoso, *More on asymptotically Anti-de Sitter spaces in topologically massive gravity*, *Phys. Rev. D* **82** (2010) 064038 [[arXiv:1006.0273](#)] [[INSPIRE](#)].
- [29] M. Kamata and T. Koikawa, *The electrically charged BTZ black hole with self (antiself) dual Maxwell field*, *Phys. Lett. B* **353** (1995) 196 [[hep-th/9505037](#)] [[INSPIRE](#)].
- [30] E.W. Hirschmann and D.L. Welch, *Magnetic solutions to $(2+1)$ gravity*, *Phys. Rev. D* **53** (1996) 5579 [[hep-th/9510181](#)] [[INSPIRE](#)].
- [31] M. Cataldo and P. Salgado, *Static Einstein-Maxwell solutions in $(2+1)$ -dimensions*, *Phys. Rev. D* **54** (1996) 2971 [[INSPIRE](#)].
- [32] M. Cataldo and P. Salgado, *Three dimensional extreme black hole with self (anti-self) dual Maxwell field*, *Phys. Lett. B* **448** (1999) 20 [[INSPIRE](#)].
- [33] M. Cataldo, *Azimuthal electric field in a static rotationally symmetric $(2+1)$ -dimensional space-time*, *Phys. Lett. B* **529** (2002) 143 [[gr-qc/0201047](#)] [[INSPIRE](#)].
- [34] M. Cataldo, J. Crisostomo, S. del Campo and P. Salgado, *On magnetic solution to $(2+1)$ Einstein-Maxwell gravity*, *Phys. Lett. B* **584** (2004) 123 [[hep-th/0401189](#)] [[INSPIRE](#)].
- [35] J. Matyjasek and O.B. Zaslavskii, *Extremal limit for charged and rotating $(2+1)$ -dimensional black holes and Bertotti-Robinson geometry*, *Class. Quant. Grav.* **21** (2004) 4283 [[gr-qc/0404090](#)] [[INSPIRE](#)].
- [36] A.A. Garcia-Diaz, *Three dimensional stationary cyclic symmetric Einstein-Maxwell solutions; black holes*, *Ann. Phys.* **324** (2009) 2004 [[arXiv:1307.6655](#)] [[INSPIRE](#)].
- [37] R.L. Arnowitt, S. Deser and C.W. Misner, *The dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997 [[gr-qc/0405109](#)] [[INSPIRE](#)].
- [38] A. J. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian systems*, RX-748 PRINT-75-0141 (1976).
- [39] M. Henneaux and C. Teitelboim, *The cosmological constant as a canonical variable*, *Phys. Lett. B* **143** (1984) 415 [[INSPIRE](#)].
- [40] C. Bunster and A. Pérez, *Superselection rule for the cosmological constant in three-dimensional spacetime*, *Phys. Rev. D* **91** (2015) 024029 [[arXiv:1412.1492](#)] [[INSPIRE](#)].
- [41] T. Regge and C. Teitelboim, *Role of surface integrals in the hamiltonian formulation of general relativity*, *Ann. Phys.* **88** (1974) 286 [[INSPIRE](#)].
- [42] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, *Black holes and asymptotics of $2+1$ gravity coupled to a scalar field*, *Phys. Rev. D* **65** (2002) 104007 [[hep-th/0201170](#)] [[INSPIRE](#)].
- [43] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, *Asymptotically Anti-de Sitter spacetimes and scalar fields with a logarithmic branch*, *Phys. Rev. D* **70** (2004) 044034 [[hep-th/0404236](#)] [[INSPIRE](#)].
- [44] M. Cardenas, O. Fuentealba and C. Martínez, *Three-dimensional black holes with conformally coupled scalar and gauge fields*, *Phys. Rev. D* **90** (2014) 124072 [[arXiv:1408.1401](#)] [[INSPIRE](#)].
- [45] A. Perez, D. Tempo and R. Troncoso, *Higher spin gravity in 3D: black holes, global charges and thermodynamics*, *Phys. Lett. B* **726** (2013) 444 [[arXiv:1207.2844](#)] [[INSPIRE](#)].
- [46] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)] [[INSPIRE](#)].

- [47] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [48] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [49] P. Kraus, *Lectures on black holes and the AdS_3/CFT_2 correspondence*, *Lect. Notes Phys.* **755** (2008) 193 [[hep-th/0609074](#)] [[INSPIRE](#)].
- [50] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Building a holographic superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601 [[arXiv:0803.3295](#)] [[INSPIRE](#)].
- [51] D. Maity, S. Sarkar, N. Sircar, B. Sathiapalan and R. Shankar, *Properties of CFTs dual to charged BTZ black-hole*, *Nucl. Phys. B* **839** (2010) 526 [[arXiv:0909.4051](#)] [[INSPIRE](#)].
- [52] J. Ren, *One-dimensional holographic superconductor from AdS_3/CFT_2 correspondence*, *JHEP* **11** (2010) 055 [[arXiv:1008.3904](#)] [[INSPIRE](#)].
- [53] T. Faulkner and N. Iqbal, *Friedel oscillations and horizon charge in 1D holographic liquids*, *JHEP* **07** (2013) 060 [[arXiv:1207.4208](#)] [[INSPIRE](#)].
- [54] G.T. Horowitz, N. Iqbal and J.E. Santos, *Simple holographic model of nonlinear conductivity*, *Phys. Rev. D* **88** (2013) 126002 [[arXiv:1309.5088](#)] [[INSPIRE](#)].
- [55] P. Chaturvedi and G. Sengupta, *Rotating BTZ black holes and one dimensional holographic superconductors*, *Phys. Rev. D* **90** (2014) 046002 [[arXiv:1310.5128](#)] [[INSPIRE](#)].
- [56] G. Barnich and P.-H. Lambert, *Einstein- Yang-Mills theory: asymptotic symmetries*, *Phys. Rev. D* **88** (2013) 103006 [[arXiv:1310.2698](#)] [[INSPIRE](#)].
- [57] M. Henneaux, *Asymptotically Anti-de Sitter universes in $D = 3, 4$ and higher dimensions*, in the proceedings of the *Fourth Marcel Grossmann Meeting on General Relativity*, R. Rufini ed., Elsevier Science Publishers (1986).